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Bernstein and Nikolskii Inequalities for Erdős Weights

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Let $W := e^{-Q}$ where Q is even, sufficiently smooth, and of faster than polynomial growth at infinity. Such a function W is often called an *Erdős weight*. In this paper we prove Nikolskii inequalities for Erdős weights. We also motivate the usefulness of, and prove a Bernstein inequality of, the form

$$\max_{x\in\mathbb{R}}\left|P'(x)\,W(x)\left|1-\left(\frac{x}{a_\beta n}\right)^2\right|^{\alpha}\right|\leqslant C\frac{n}{a_n}\max_{x\in\mathbb{R}}\left|P(x)\,W(x)\left|1-\left(\frac{x}{a_\beta n}\right)^2\right|^{\alpha-1/2}\right|,$$

for fixed $\alpha\geqslant\frac{1}{2}$, $\beta>1$, $P\in\mathscr{P}_n$, n large enough and $C>0$ independent of n, P, and $x\in\mathbb{R}$. Here, a_n is the nth Mhaskar–Rahmanov–Saff number for W. © 1993 Academic Press, Inc.

1. Introduction and Statement of Results

In recent years, attention has been given to Christoffel function estimates and L_∞ Markov–Bernstein inequalities for Erdős weights. See [4–6]. The extension of Markov–Bernstein inequalities to L_p requires the use of Nikolskii inequalities since Nikolskii inequalities give a relationship between metrics in different finite dimensional metric spaces of polynomials. Our Bernstein inequalities will be useful in the study of rates of polynomial approximation. Some of the ideas of proof of sharpness of the Nikolskii inequalities are those of Nevai and Totik [10]. The proof of our Bernstein inequalities uses results of Lubinsky [4, 5]. Christoffel function estimates established by Lubinsky and Mthembu [6] are crucial ingredients of these proofs.

In this section we state our main results. We prove Nikolskii inequalities and Bernstein inequalities in Sections 3 and 4, respectively.

Throughout, $\mathscr{P}_n$, $n = 1, 2, 3, \ldots$, denotes the class of real polynomials of degree at most n. Further, $C, C_1, C_2, \ldots$, denote positive constants independent of n, $P\in\mathscr{P}_n$, and $x\in\mathbb{R}$, which are not necessarily the same from line to line. We use the usual o, O, notation and $\sim$ as in [3–6]: We write

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$f(x) \sim g(x)$ if there exist C_1, C_2 with $C_1 \leqslant f(x)/g(x) \leqslant C_2$ for the specified range of x. Similar notation is used for sequences.

The classical inequalities of Markov and Bernstein are respectively

$$\|P'\|_{[-1,1]} \leqslant n^2 \|P\|_{[-1,1]}, \qquad P \in \mathscr{P}_n, \tag{1.1}$$

and

$$|P'(x)| \leqslant n(1-x^2)^{-1/2} \|P\|_{[-1,1]}, \qquad P \in \mathscr{P}_n, |x| < 1. \tag{1.2}$$

The interest in these inequalities lies in their application to rates of approximation by polynomials. Their weighted analogues are used similarly on rates of approximation by weighted polynomials. The most general analogue of (1.1) for Erdős weights appeared in [4]. We need an analogue of (1.2) which will be useful in establishing convergence of orthogonal expansions associated with Erdős weights.

To state our results we need some notation:

DEFINITION 1.1. Let $W := e^{-Q}$, where Q is even and continuous in $\mathbb{R}$, Q''' exists in $(0, \infty)$, and Q' is positive in $(0, \infty)$. Let

$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}, \qquad x \in (0, \infty), \tag{1.3}$$

be increasing in $(0, \infty)$, with

$$\lim_{x \to 0^+} T(x) = T(0^+) > 1, \tag{1.4}$$

$$\lim_{x \to \infty} T(x) = \infty, \tag{1.5}$$

and for each $\varepsilon > 0$,

$$T(x) = O(Q'(x)^\varepsilon), \qquad x \to \infty. \tag{1.6}$$

Assume further that

$$\frac{Q''(x)}{Q'(x)} \sim \left\{\frac{Q'(x)}{Q(x)}\right\}, \qquad x \text{ large enough}, \tag{1.7}$$

and for some $C > 0$,

$$\frac{|Q'''(x)|}{Q'(x)} \leqslant C \left\{\frac{Q'(x)}{Q(x)}\right\}^2, \qquad x \text{ large enough}. \tag{1.8}$$

Then we say that W is an *Erdős weight of class* 3, and we write $W \in SE^*(3)$.

Remarks. (a) The limit (1.5) implies that $Q(x)$ grows faster than any polynomial at infinity, while (1.6) is a weak regularity condition: one typically has [4, 5]

$$T(x) = O([\log Q'(x)]^{1+\varepsilon}), \qquad x \to \infty, \tag{1.9}$$

for each $\varepsilon > 0$. The restriction (1.4) simplifies analysis.

(b) The class $SE^*(3)$ is contained in the class $SE(3)$ of [5], for in [5] we take only $\varepsilon = \frac{1}{15}$ in (1.6).

(c) As examples of $W \in SE^*(3)$ we mention

$$W(x) := \exp(-\exp_k(|x|^\alpha)), \qquad x \in \mathbb{R}, \alpha > 1, k \text{ is a positive integer}, \tag{1.10}$$

where $\exp_k$ denotes the kth iterated exponential $\exp(\exp \ldots)$ (k times). Another example is

$$W(x) := \exp(-\exp\{\log(A + x^2)\}^\alpha), \qquad x \in \mathbb{R}, \alpha > 1, A \text{ large enough}. \tag{1.11}$$

DEFINITION 1.2. Let $W := e^{-Q(x)}$, where $Q(x)$ is even and continuous in $\mathbb{R}$, $Q'(x)$ exists in $(0, \infty)$, and $xQ'(x)$ is increasing in $(0, \infty)$ with limits 0 and ∞ at 0 and ∞, respectively. For $u > 0$, we define the *Mhaskar–Rahmanov–Saff number* $a_u = a_u(W)$ to be the positive root of the equation

$$u := \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t)(1 - t^2)^{-1/2} \, dt. \tag{1.12}$$

It follows easily from the conditions of Definition 1.2 that for all $u > 0$, a_u exists and is unique. The number a_n, $n = 1, 2, 3, \ldots$, is very important in that the suprenum norm of a weighted polynomial lives in $[-a_n, a_n]$ (see [7]).

DEFINITION 1.3. Given p and q such that $0 < p$, $q \leqslant \infty$, define the Nikolskii constant $N_n := N_n(p, q)$, $n = 1, 2, 3, \ldots$, by

$$N_n(p, q) := \begin{cases} a_n^{1/p - 1/q}, & \text{if } p \leqslant q \\ [(n/a_n)\, T(a_n)^{1/2}]^{1/q - 1/p}, & \text{if } p > q. \end{cases} \tag{1.13}$$

We are now ready to state our main results.

THEOREM 1.4. (Nikolskii Inequality). *Let $W \in SE^*(3)$, a_n be as in Definition* 1.2, *and* $0 < p$, $q \leqslant \infty$. *Then there exists $C > 0$ such that for $P \in \mathcal{P}_n$ and n large enough,*

$$\|PW\|_{L_p(\mathbb{R})} \leqslant C N_n \|PW\|_{L_q(\mathbb{R})}. \tag{1.14}$$

In Section 3, we prove (1.14) sharp for $p \leqslant q$ and also sharp for $p = \infty$ and $q = 2$; and finally for $2 < q < p < \infty$.

THEOREM 1.5. *Let* $W \in SE^*(3)$,

$$\varphi_n(x) := |1 - x^2| + 1/T(a_n), \qquad x \in \mathbb{R}. \tag{1.15}$$

Let β, $\Delta > 0$ *and* $\alpha \geqslant \frac{1}{2}$. *Then there exists* $C > 0$ *such that for* $P \in \mathscr{P}_n$ *and* n *large enough,*

$$\max_{x \in \mathbb{R}} \left| P'(x)\, W(x)\, \varphi_n\left(\frac{x}{a_{\beta n}}\right)^{\alpha} \right| \leqslant C \frac{n}{a_n} \max_{x \in \mathbb{R}} \left| P(x)\, W(x)\, \varphi_n\left(\frac{x}{a_{\Delta n}}\right)^{\alpha - 1/2} \right|. \tag{1.16}$$

THEOREM 1.6. (Bernstein Inequality). *Let* $W \in SE^*(3)$. *Let* $\alpha \geqslant \frac{1}{2}$ *and* $\beta > 1$. *Then there exists* $C > 0$ *such that for* $P \in \mathscr{P}_n$ *and* n *large enough,*

$$\max_{x \in \mathbb{R}} \left| P'(x)\, W(x) \left| 1 - \left(\frac{x}{a_{\beta n}}\right)^2 \right|^{\alpha} \right| \leqslant C \frac{n}{a_n} \max_{x \in \mathbb{R}} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{\beta n}}\right)^2 \right|^{\alpha - 1/2} \right|. \tag{1.17}$$

Remarks. (a) The Markov inequality in [5, Theorem 2.6, p. 15] reads

$$\|P'W\|_{L_\infty(\mathbb{R})} \leqslant C \frac{n}{a_n} T(a_n)^{1/2} \|PW\|_{L_\infty(\mathbb{R})}, \qquad P \in \mathscr{P}_n. \tag{1.18}$$

In particular, it is valid for $W \in SE^*(3)$. Furthermore, the dependence on n—namely $(n/a_n)\, T(a_n)^{1/2}$—is sharp (See [5, Theorem 2.6]) and is $O(n^2)$ (cf. (1.1) above).

(b) The function φ_n defined by (1.15) plays much the same role for Erdős weights as does the factor $(\sqrt{1-x} + 1/n)(\sqrt{1+x} + 1/n)$ in analogous questions for weights on $[-1, 1]$. See [9, Theorem 9.19, p. 164].

(c) Theorem 4.1 below shows that for $\alpha = \frac{1}{2}$, (1.17) is valid even for $\beta = 1$.

(d) Without the factor $(1 - (x/a_{\beta n})^2)^{\alpha}$ in (1.17) above, a factor $T(a_n)^{1/2}$ would appear on the right-hand side of (1.17).

(e) If, for example, $\alpha > 0$, k is a positive integer and (see (1.10))

$$W(x) = e^{-Q(x)},$$

where

$$Q(x)=\exp_k(|x|^{\alpha}), \qquad x\in\mathbb{R}, \tag{1.19}$$

then all conditions of Definition 1.1 are satisfied and

$$T(x)=\alpha\left\{\prod_{j=1}^{k}\log_j Q(x)\right\}(1+o(1)), \qquad x\to\infty. \tag{1.20}$$

A lengthy computation involving Laplace's method shows that

$$a_n=(\log_k n)^{1/\alpha}(1+o(1)), \tag{1.21}$$

and

$$\begin{aligned} Q'(a_n) &\sim \frac{n}{a_n}T(a_n)^{1/2} \\ &\sim n\left[\prod_{j=1}^{k}\log_j n\right]^{1/2}(\log_k n)^{-1/\alpha}, \qquad n\to\infty. \end{aligned} \tag{1.22}$$

Thus, for the weight above, (1.14), (1.16), and (1.17) become, respectively:

$$\|PW\|_{L_p(\mathbb{R})} \leqslant C\begin{cases}[(\log_k n)^{1/\alpha}]^{1/p-1/q}, & p\leqslant q\\ [n(\prod_{j=1}^{k}\log_j n)^{1/2}(\log_k n)^{-1/\alpha}]^{1/q-1/p}, & p>q\end{cases}\|PW\|_{L_q(\mathbb{R})}; \tag{1.14'}$$

$$\max_{x\in\mathbb{R}}\left|(P'W)(x)\,\varphi_n\left(\frac{x}{a_{\beta n}}\right)^{\alpha}\right| \leqslant Cn(\log_k)^{-1/\alpha}\max_{x\in\mathbb{R}}\left|(PW)(x)\,\phi_n\left(\frac{x}{a_{\Delta n}}\right)^{\alpha-1/2}\right|, \tag{1.16'}$$

where

$$\varphi_n(x):=|1-x^2|+1\Big/\prod_{j=1}^{k}\log_j n;$$

$$\max_{x\in\mathbb{R}}\left|(P'W)(x)\left|1-\left(\frac{x}{a_{\beta n}}\right)^2\right|^{\alpha}\right| \leqslant Cn(\log_k n)^{-1/\alpha}\max_{x\in\mathbb{R}}\left|(PW)(x)\left|1-\left(\frac{x}{a_{\beta n}}\right)^2\right|^{\alpha-1/2}\right|. \tag{1.17'}$$

2. Preliminary Results

Let $W\colon \mathbb{R} \to \mathbb{R}$ be even, positive, and continuous, and such that all power moments

$$\int_{-\infty}^{\infty} x^j W(x)\,dx, \qquad j=0, 1, 2, \ldots,$$

exist. Associated with W^2 are the orthonormal polynomials p_j of degree j, $j=0, 1, 2, \ldots$, satisfying

$$\int_{-\infty}^{\infty} p_j(x)\,p_k(x)\,W^2(x)\,dx=\delta_{jk}, \qquad j, k=0, 1, 2, \ldots. \tag{2.1}$$

The nth *Christoffel function* is

$$\lambda_n(W^2, x) := \inf_{P\in\mathscr{P}_{n-1}} \int_{-\infty}^{\infty} (PW)^2(t)\,dt/P^2(x)=\left\{\sum_{j=0}^{n-1} p_j^2(x)\right\}^{-1}, \tag{2.2}$$

$x\in\mathbb{R}$, $n=1, 2, 3, \ldots$. Given a fixed $s\geqslant 0$, we shall often use the notation

$$c_n := a_n(1+s\Delta_n), \tag{2.3}$$

where $\Delta_n=((\log n)/nT(a_n))^{2/3}$.

Lemma 2.1. *Let $W\in SE^*(3)$, $0<p<\infty$, and c_n be defined by* (2.3), *with $s\geqslant 0$ fixed but large enough. Then for $P\in\mathscr{P}_n$ and n large enough,*

$$\|PW\|_{L_p(\mathbb{R})}\leqslant 2\,\|PW\|_{L_p[-c_n, c_n]}. \tag{2.4}$$

Moreover, if $p=\infty$, we have

$$\|PW\|_{L_\infty(\mathbb{R})}=\|PW\|_{L_\infty[-a_n, a_n]}. \tag{2.5}$$

Proof. Equation (2.4) follows from (5.8) of Theorem 5.2 in [5], and (2.5) is (5.1) of Theorem 5.1 in [5]. ∎

Lemma 2.2. *Let $W\in SE^*(3)$, and let a_n, $n\geqslant 1$, denote the nth Mhaskar–Rahmanov–Saff number for Q, defined by* (1.12). *Then for n large enough,*

$$\max_{x\in\mathbb{R}} \lambda_n^{-1}(W^2, x)\,W^2(x)\sim\frac{n}{a_n}\,T(a_n)^{1/2}, \tag{2.6}$$

and

$$\max_{x\in\mathbb{R}} \lambda_n^{-1}(W^2, x)\,W^2(x)\left|1-\left(\frac{x}{a_n}\right)^{1/2}\right|\sim\frac{n}{a_n}. \tag{2.7}$$

Proof. These are respectively (1.17) and (1.18) of Theorem 1.2 in [6]. ∎

LEMMA 2.3 *Let $W \in SE^*(3)$. For any fixed $0 < \alpha < \beta < \infty$, as $n \to \infty$,*

$$T(a_{\alpha n}) \sim T(a_{\beta n}). \tag{2.8}$$

Furthermore,

$$1 - \frac{a_{\alpha n}}{a_{\beta n}} \sim T(a_n)^{-1}, \tag{2.9}$$

$$\lim_{u \to \infty} \frac{a_{\alpha u}}{a_u} = 1, \tag{2.10}$$

and for each $\varepsilon > 0$,

$$T(a_n) = O(n^{\varepsilon}), \qquad n \to \infty. \tag{2.11}$$

Proof. Equations (2.8) and (2.9) are respectively (2.7) and (2.8) of Lemma 2.2 in [6]. Equation (2.10) is (3.18) of Lemma 3.2 in [5]. Equation (2.11) follows from (1.6) and (2.25) of Lemma 2.3 in [4], noting that the function $\chi(x)$ there is $T(x)$ in this paper, and that $a_n \to \infty$ as $n \to \infty$. ∎

We observe that if $W \in SE^*(3)$, then

$$W^k := e^{-kQ} \in SE^*(3), \qquad k = 1, 2, 3, \ldots.$$

This is easy to show because $T(x)$, defined by (1.3), is the same for both W and W^k.

LEMMA 2.4. *Let $W \in SE^*(3)$, and let a_n, $n \geq 1$, be defined by* (1.12). *Then for n large enough,*

$$\max_{x \in \mathbb{R}} \lambda_n^{-1}(W^{2k}, x)\, W^{2k}(x) \sim \frac{n}{a_n}\, T(a_n)^{1/2}, \qquad k = 1, 2, 3, \ldots. \tag{2.12}$$

Proof. We note that the nth Mhaskar–Rahmanov–Saff number for kQ, $k = 1, 2, 3, \ldots$, which we denote by a_n^*, is the positive root of

$$n = \frac{2}{\pi} \int_0^1 a_n^* t (kQ)'(a_n^* t)(1 - t^2)^{-1/2}\, dt.$$

Then, the above implies that $a_n^* = a_{n/k}$. We obtain (2.12) from (2.6), (2.8), and the observation before this lemma. ∎

LEMMA 2.5. *Let* $W \in SE^*(3)$. *For* $n \geq 1$, *let*

$$\psi_n(x) := \int_{1/(2a_n)}^{1} (1-s)^{-1/2} \frac{a_n x Q'(a_n x) - a_n s Q'(a_n s)}{a_n x - a_n s}\, ds, \qquad x \in [0, 1]. \tag{2.13}$$

Then there exists $C > 0$ *such that for* $P \in \mathscr{P}_n$ *and* n *large enough,*

$$|(PW)'(x)| \leq C \|PW\|_{L_\infty(\mathbb{R})} \times \begin{cases} (1-|x|/a_n)^{-1} \int_{|x|/a_n}^{1} \psi_n(t)(1-t)^{1/2}\, dt, & |x| \leq a_n[1-(nT(a_n))^{-2/3}] \\ (nT(a_n))^{2/3}/a_n, & |x| \geq a_n[1-(nT(a_n))^{-2/3}]. \end{cases} \tag{2.14}$$

Proof. This follows from (1.25) of Theorem 1.5 in [4], with $\varepsilon = \frac{1}{2}$, $r = 1$, and the fact that A_n^* in [4] is such that $A_n^* \sim T(a_n)$ [5, (3.44) of Lemma 3.4]. ∎

LEMMA 2.6. *Let* $W \in SE^*(3)$. *Let*

$$\mu_{n, a_n}(x) := \frac{2}{\pi^2} \int_0^1 \left(\frac{1-x^2}{1-s^2}\right)^{1/2} \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{n(s^2 - x^2)}\, ds, \tag{2.15}$$

$x \in (-1, 1)$, $n \geq 1$. *Then for* n *large enough and* $\frac{1}{2} \leq t \leq 1$,

$$\mu_{n, a_n}(t) \sim (1-t)^{1/2} \frac{a_n}{n} \psi_n(t), \tag{2.16}$$

and

$$\mu_{n, a_n}(t)(1-t)^{1/2} \leq C. \tag{2.17}$$

Proof. Equation (2.16) follows from (3.26) of Lemma 3.2 in [4], with $\varepsilon = \frac{1}{2}$. Equation (2.17) follows from (2.14) of Theorem 2.4 in [6]. ∎

3. PROOF OF THE NIKOLSKII INEQUALITY

LEMMA 3.1. *Under the conditions of Theorem* 1.4, *with* $p = \infty$ *and* q *arbitrary*, (1.14) *holds.*

Proof. We prove this lemma in three steps. We can express an arbitrary $P \in \mathscr{P}_n$ in the form

$$P(x) = \sum_{j=0}^{n} d_j p_j(W^2, x).$$

By the Cauchy–Schwarz inequality and Parseval's identity,

$$|(PW)(x)| \leqslant \left(\sum_{j=0}^{n} d_j^2 \right)^{1/2} \left(\sum_{j=0}^{n} p_j^2(W^2, x)\, W^2(x) \right)^{1/2}$$
$$= \left(\int |(PW)(x)|^2\, dx \right)^{1/2} (\lambda_{n+1}^{-1}(W^2, x)\, W^2(x))^{1/2}.$$

If we take the maximum over $\mathbb{R}$ on the above, and use (2.6) and (2.8), (1.14) holds for $p = \infty$ and $q = 2$.

Next, we show that (1.14) holds for $p = \infty$ and $q = 2k$, $k = 1, 2, 3, \ldots$. We note that $P^k \in \mathscr{P}_{nk}$, $k \geqslant 1$, whenever $P \in \mathscr{P}_n$. By (2.8), (2.12), and the inequality above,

$$\|P^k W^k\|_{L_\infty(\mathbb{R})} \leqslant \|P^k W^k\|_{L_2(\mathbb{R})} (\max_{x \in \mathbb{R}} \lambda_{nk+1}^{-1}(W^{2k}, x)\, W^{2k}(x))^{1/2}$$
$$\leqslant C_1 \left[\frac{n}{a_n} T(a_n)^{1/2} \right]^{1/2} \|P^k W^k\|_{L_2(\mathbb{R})}.$$

Then the result follows from the above by taking kth roots on both sides. Finally, we show that (1.14) holds for $p = \infty$ and q arbitrary.

Let $2k$ be the smallest positive integer greater than or equal to q, q fixed but arbitrary. Then since (1.14) holds for $p = \infty$ and $q = 2k$, $k \geqslant 1$, we have

$$\|PW\|_{L_\infty(\mathbb{R})} \leqslant C_2 \left[\frac{n}{a_n} T(a_n)^{1/2} \right]^{1/2k} \left(\int |(PW)(x)|^q\, |(PW)(x)|^{2k-q}\, dx \right)^{1/2k}$$
$$\leqslant C_3 \left[\frac{n}{a_n} T(a_n)^{1/2} \right]^{1/2k} \|PW\|_{L_\infty(\mathbb{R})}^{(2k-q)/2k} \left(\int |(PW)(x)|^q \right)^{1/2k}.$$

So,

$$\|PW\|_{L_\infty(\mathbb{R})}^{q/2k} \leqslant C_3 \left[\frac{n}{a_n} T(a_n)^{1/2} \right]^{1/2k} \|PW\|_{L_q(\mathbb{R})}^{q/2k}.$$

This completes our proof. ∎

Proof of (1.14) *of Theorem* 1.4 *for* $p \leqslant q$. By Hölder's inequality and (2.4) above,

$$\|PW\|_{L_p(\mathbb{R})} \leqslant 2 \left[\int_{-c_n}^{c_n} |(PW)(x)|^{pq/p}\, dx \right]^{1/q} \left[\int_{-c_n}^{c_n} dx \right]^{(q-p)/pq}$$
$$\leqslant C_4 \|PW\|_{L_q(\mathbb{R})}\, c_n^{1/p - 1/q}.$$

Then (1.14) follows for $p \leqslant q$ from the above since $c_n \leqslant C_5 a_n$ as $\Delta_n = o(1)$ in (2.3). ∎

Proof of (1.14) *of Theorem* 1.4 *for* $p>q$. By Lemma 3.1 above,

$$\|PW\|_{L_p(\mathbb{R})}=\left(\int |(PW)(x)|^q\,|(PW)(x)|^{p-q}\,dx\right)^{1/p}$$
$$\leqslant \|PW\|_{L_\infty(\mathbb{R})}^{(p-q)/p}\,\|PW\|_{L_q(\mathbb{R})}^{q/p}$$
$$\leqslant C_6\left[\frac{n}{a_n}\,T(a_n)^{1/2}\right]^{(p-q)/pq}\|PW\|_{L_q(\mathbb{R})}^{(p-q)/p}\,\|PW\|_{L_q(\mathbb{R})}^{q/p}.$$

Then (1.14) for $p>q$ follows from the inequality above. ∎

Using the methods of [10], we now investigate the sharpness of Theorem 1.6.

Proof of the Sharpness of (1.14) *of Theorem* 1.4 *for* $p\leqslant q$. We show that under the conditions of Theorem 1.4 there exist a constant $C>0$ and a sequence of polynomials $\{S_n^*\}_{n=1}^{\infty}$ with degree $S_n^*\leqslant n$, such that

$$\|S_n^*W\|_{L_p(\mathbb{R})}\geqslant CN_n\,\|S_n^*W\|_{L_q(\mathbb{R})}. \tag{3.1}$$

Under our conditions on Q, Theorem 5.4 in [1] implies the existence of an even entire function G defined by

$$G(x):=\sum_{j=0}^{\infty} h_{2j}x^{2j},\qquad h_{2j}\geqslant 0, \tag{3.2}$$

such that

$$G(x)\sim W(x)^{-1},\qquad x\in\mathbb{R}.$$

We define S_n^* to be the $\langle n/2\rangle$nd partial sum of the power series in (3.2) above. Then

$$0\leqslant S_n^*(x)\leqslant C_7\,W(x)^{-1},\qquad x\in\mathbb{R}. \tag{3.3}$$

By the Hermite contour integral error formula,

$$G(x)-S_n^*(x)=\frac{1}{2\pi i}\int_{|t|=r}\frac{G(t)}{t-x}\left(\frac{x}{t}\right)^{n+1}dt,\qquad |x|<r$$

and so

$$|G(x)-S_n^*(x)|\leqslant\frac{1}{2\pi}\,2\pi r\max_{|t|=r}\left|\frac{G(t)}{t-x}\right|\left|\frac{x}{r}\right|^{n+1}$$
$$\leqslant r\,\frac{G(r)}{r-|x|}\left(\frac{|x|}{r}\right)^{n+1}\qquad\text{(as } G \text{ has non-negative coefficients)}$$
$$\leqslant Ce^{Q(r)}\left(\frac{|x|}{r}\right)^{n+1},\qquad\text{provided } |x|\leqslant\frac{r}{2}.$$

Now, let us suppose that for some $0<\varepsilon<\frac{1}{2}$, $|x|\leqslant \varepsilon r$. Then

$$\begin{aligned}|G(x)-S_n^*(x)| &\leqslant C_1 \exp(Q(r))\,\varepsilon^n\\ &= C_1 \exp\left(n\left[-\log\frac{1}{\varepsilon}+\frac{Q(r)}{n}\right]\right),\end{aligned}$$

and choosing $r=a_n$, we get

$$|G(x)-S_n^*(x)| \leqslant C_1 \exp\left(n\left[-\log\frac{1}{\varepsilon}+\frac{Q(a_n)}{n}\right]\right)=o(1),$$

since $Q(a_n)=o(n)$, $n\to\infty$ (see [4, Lemma 2.2(a)]). This shows that there exists $0<\varepsilon<\frac{1}{2}$ such that

$$G(x)\leqslant S_n^*(x)+o(1),$$

uniformly for $|x|\leqslant \varepsilon a_n$ and n large enough. Thus we have

$$W(x)^{-1}\leqslant C_8 S_n^*(x), \qquad |x|\leqslant \varepsilon a_n. \tag{3.4}$$

Given $r>0$, by (3.3) and (3.4) we have

$$\|S_n^* W\|_{L_r(\mathbb{R})}\geqslant \|S_n^* W\|_{L_r[-\varepsilon a_n,\,\varepsilon a_n]}\sim a_n^{1/r}, \tag{3.5}$$

and by (2.4)

$$\|S_n^* W\|_{L_r(\mathbb{R})}\leqslant 2\,\|S_n^* W\|_{L_r[-c_n,\,c_n]}\leqslant C_9 a_n^{1/r}. \tag{3.6}$$

So, for any $0<r<\infty$,

$$\|S_n^* W\|_{L_r(\mathbb{R})}\sim a_n^{1/r}, \qquad n\geqslant 1,$$

and hence (3.1). ▌

Proof of the Sharpness of (1.14) *of Theorem* 1.4 *for* $p=\infty$ *and* $q=2$. By (2.2),

$$\sup_{x\in\mathbb{R}} \lambda_n^{-1}(W^2,x)\,W^2(x)\leqslant \sup_{P\in\mathscr{P}_{n-1}} \frac{\|PW\|^2_{L_\infty(\mathbb{R})}}{\|PW\|^2_{L_2(\mathbb{R})}}.$$

Using (2.11) and taking square roots we have

$$C_{10}\left[\frac{n}{a_n}\,T(a_n)^{1/2}\right]^{1/2}\leqslant \sup_{P\in\mathscr{P}_n}\frac{\|PW\|_{L_\infty(\mathbb{R})}}{\|PW\|_{L_2(\mathbb{R})}},$$

which completes this proof. ▌

We now use the sharpness of (1.14) of Theorem 1.4 ($p=\infty$ and $q=2$) to prove sharpness for $2<q<p<\infty$.

Proof of the Sharpness of (1.14) *of Theorem* 1.4 *for* $2<q<p<\infty$. By the sharpness for $p=\infty$ and $q=2$, we have

$$\|P^*W\|_{L_\infty(\mathbb{R})}=N_n(\infty,2)\,\|P^*W\|_{L_2(\mathbb{R})}, \tag{3.7}$$

for some $P^*\in\mathscr{P}_n$. Here, $N_n(\infty,2):=C[(n/a_n)\,T(a_n)^{1/2}]^{1/2}$ (cf. (1.13)). Let $2<q<p<\infty$. By Theorem 1.4,

$$\|P^*W\|_{L_\infty(\mathbb{R})}\leqslant N_n(\infty,2)^{2/q}\,\|P^*W\|_{L_q(\mathbb{R})}. \tag{3.8}$$

Next, we observe that

$$\begin{aligned}\|P^*W\|_{L_q(\mathbb{R})}&\leqslant \|P^*W\|_{L_\infty(\mathbb{R})}^{(q-2)/q}\,\|P^*W\|_{L_2(\mathbb{R})}^{2/q}\\ &=\|P^*W\|_{L_\infty(\mathbb{R})}^{1-2/q}N_n(\infty,2)^{-2/q}\,\|P^*W\|_{L_\infty(\mathbb{R})}^{2/q} \qquad \text{(by (3.7))}\\ &=N_n(\infty,2)^{-2/q}\,\|P^*W\|_{L_\infty(\mathbb{R})}.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) we obtain

$$\|P^*W\|_{L_\infty(\mathbb{R})}=N_n(\infty,2)^{2/q}\,\|P^*W\|_{L_q(\mathbb{R})}. \tag{3.10}$$

Again, by Theorem 1.4,

$$\|P^*W\|_{L_\infty(\mathbb{R})}\leqslant N_n(\infty,2)^{2/p}\,\|P^*W\|_{L_p(\mathbb{R})},$$

and so

$$\begin{aligned}\|P^*W\|_{L_q(\mathbb{R})}&\leqslant N_n(\infty,2)^{-2/q}\,\|P^*W\|_{L_\infty(\mathbb{R})}\\ &=N_n(\infty,2)^{-(2/q-2/p)}\,\|P^*W\|_{L_p(\mathbb{R})},\end{aligned} \tag{3.11}$$

by (3.10). This completes our proof. ∎

4. Proof of the Bernstein Inequality

THEOREM 4.1. *Let $W\in SE^*(3)$. Then there exists $C>0$ such that for $P\in\mathscr{P}_n$ and n large enough,*

$$\max_{x\in\mathbb{R}}\left|P'(x)\,W(x)\left|1-\left(\frac{x}{a_n}\right)^2\right|^{1/2}\right|\leqslant C\,\frac{n}{a_n}\,\|PW\|_{L_\infty(\mathbb{R})}. \tag{4.1}$$

Proof. $|(PW)'(x)|=|(P'W)(x)+Q'(x)(PW)(x)|$. So,

$$|(P'W)(x)|\leqslant|(PW)'(x)|+|Q'(x)(PW)(x)|.$$

It is easy to show that

$$1-\left(\frac{x}{a_n}\right)^2 \sim 1-\frac{|x|}{a_n}, \qquad |x| \leqslant a_n.$$

By (2.14), (2.16), and the two inequalities above, and for $a_n/2 \leqslant |x| \leqslant a_n[1-(nT(a_n))^{-2/3}]$,

$$\begin{aligned}|(P'W)(x)| &\left(1-\left(\frac{x}{a_n}\right)^2\right)^{1/2} \\ &\leqslant C_{11}\, \|PW\|_{L_\infty(\mathbb{R})} \left(1-\left(\frac{x}{a_n}\right)^2\right)^{-1/2} \int_{|x|/a_n}^{1} \frac{n}{a_n}\, \mu_{n,a_n}(t)\, dt \\ &\quad + |Q'(x)| \left(1-\left(\frac{x}{a_n}\right)^2\right)^{1/2} \|PW\|_{L_\infty(\mathbb{R})}. \end{aligned} \tag{4.2}$$

We estimate the first term on the right-hand side of (4.2). By (2.17)

$$\int_{|x|/a_n}^{1} \mu_{n,a_n}(t)\, dt \leqslant C_{12} \int_{|x|/a_n}^{1} \frac{dt}{\sqrt{1-t^2}} \leqslant C_{13} \left(1-\frac{|x|}{a_n}\right)^{1/2}. \tag{4.3}$$

To estimate the second term of (4.2), we recall that a_n, where n is a positive integer, is the positive root of

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t)(1-t^2)^{-1/2}\, dt.$$

So,

$$\begin{aligned} n &\geqslant \frac{2}{\pi} \int_{|x|/a_n}^{1} a_n t Q'(a_n t)(1-t^2)^{-1/2}\, dt \\ &\geqslant \frac{2}{\pi} \int_{|x|/a_n}^{1} x Q'(x)(1-t^2)^{-1/2}\, dt \\ &\geqslant C_{14} a_n\, |Q'(x)| \left(1-\frac{|x|}{a_n}\right)^{1/2}. \end{aligned} \tag{4.4}$$

By (4.2)–(4.4), and for $a_n/2 \leqslant |x| \leqslant a_n[1-(nT(a_n))^{-2/3}]$,

$$\left|(P'W)(x)\left(1-\left(\frac{x}{a_n}\right)^2\right)^{1/2}\right| \leqslant C_{15} \frac{n}{a_n}\, \|PW\|_{L_\infty(\mathbb{R})}. \tag{4.5}$$

For $a_n[1-(nT(a_n))^{-2/3}] \leqslant |x| \leqslant a_n$, the only difference is that by (2.13) and (2.11),

$$|(PW)'(x)| \left(1-\left(\frac{x}{a_n}\right)^2\right)^{1/2} \leqslant C_{16}(nT(a_n))^{2/3}/a_n \, \|PW\|_{L_\infty(\mathbb{R})}$$

$$\leqslant C_{17} \frac{n}{a_n} \|PW\|_{L_\infty(\mathbb{R})}.$$

We close the gap by showing that (4.5) holds even for $|x| \leqslant a_n/2$. This follows from Corollary 3.2 in [3]. It is easy to show that ξ_x in [3], the root of $\xi_x^2 Q''(\xi_x) = x$, x large, satisfies $\xi_n/a_n \to 1$, $n \to \infty$. So we have shown

$$\max_{[-a_n, a_n]} \left|(P'W)(x)\left(1-\left(\frac{x}{a_n}\right)^2\right)^{1/2}\right| \leqslant C \frac{n}{a_n} \|PW\|_{L_\infty(\mathbb{R})}.$$

As a_{2n} for W^2 is a_n for W, we have

$$\max_{x\in\mathbb{R}} \left|P'(x)^2\left(1-\left(\frac{x}{a_n}\right)^2\right) W^2(x)\right| = \max_{[-a_n, a_n]} \left|P'(x)^2\left(1-\left(\frac{x}{a_n}\right)^2\right) W^2(x)\right|,$$

and the result follows. ∎

LEMMA 4.2. *Fix $\alpha = \gamma + j$, $0 \leqslant \gamma < 1$ and $j = 0, 1, 2, \ldots$. Let*

$$u(x) := (1-x^2)^{-\gamma}, \qquad x \in [-1, 1], \tag{4.6}$$

and let

$$R_n(x) := \frac{1}{n} \lambda_n^{-1}\left(u, \frac{x}{a_{4n}}\right)\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^j. \tag{4.7}$$

Then there exists $C > 0$ such that for n large enough, and uniformly for $|x| \leqslant a_{4n}(1-n^{-2})$,

$$R_n(x) \sim \left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^{\alpha-1/2}, \tag{4.8}$$

$$|R_n'(x)| \leqslant \frac{C}{a_{4n}} \left|1-\left(\frac{x}{a_{4n}}\right)^2\right|^{\alpha-3/2}. \tag{4.9}$$

Furthermore, uniformly for $|x| \leqslant a_{3n}$,

$$\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^{\alpha-1} \leqslant CT(a_n)^{1/2}\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^{\alpha-1/2}. \tag{4.10}$$

Proof. We first show (4.8) and (4.9) for α such that $j=0$ and then extend that result to arbitrary $j=1, 2, 3, \ldots$. Let

$$u_n(x) := (\sqrt{1-x}+1/n)^{1-2\gamma}(\sqrt{1+x}+1/n)^{1-2\gamma}. \tag{4.11}$$

For $|x| \leqslant a_{4n}(1-n^{-2})$,

$$u_n\left(\frac{x}{a_{4n}}\right) \sim \left(\sqrt{1-\left(\frac{x}{a_{4n}}\right)^2}\right)^{1-2\gamma}.$$

So, by Lemma 6.3.5 in [9, p.108],

$$\lambda_n^{-1}\left(u, \frac{x}{a_{4n}}\right) \sim \frac{n}{u_n(x/a_{4n})} \sim n\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^{\gamma-1/2}, \tag{4.12}$$

and (4.8) follows when $j=0$. By (4.12) and (23) in [11, p. 36],

$$\left|\left(\lambda_n^{-1}\left(u, \frac{x}{a_{4n}}\right)\right)'\right| = \frac{|\lambda_n'(u, x/a_{4n})|}{a_{4n}\lambda_n^2(u, x/a_{4n})} \leqslant C_{18}\frac{n}{a_{4n}}\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^{\gamma-3/2},$$

and (4.9) follows when $j=0$.

It is easy to show that (4.8) for $j=1, 2, 3, \ldots$ follows from (4.8) with $j=0$. By (4.7), we have

$$\begin{aligned} R_n'(x) &= \frac{1}{na_{4n}}\left(\lambda_n^{-1}\left(u, \frac{x}{a_{4n}}\right)\right)'\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^j \\ &\quad -2j\frac{x}{a_{4n}^2}\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^{-1}\frac{1}{n}\lambda_n^{-1}\left(u, \frac{x}{a_{4n}}\right)\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^j, \end{aligned}$$

and (4.9) for $j=1, 2, 3, \ldots$ follows from (4.8) for $j=0$ and (4.9) for $j=0$. To show (4.10), we note that for $|x| \leqslant a_{3n}$,

$$1-\frac{|x|}{a_n} \geqslant 1-\frac{a_{3n}}{a_{4n}} \geqslant C_{19}/T(a_n).$$

So,

$$\left(1-\left(\frac{x}{a_{4n}}\right)^2\right)^{1/2} \geqslant C_{20}/T(a_n)^{1/2},$$

and (4.10) follows. ∎

We remark that Lemma 4.2 is also valid for $\alpha=\gamma+j$, where $j=-1, -2, -3, \ldots$.

LEMMA 4.3. *Let $W \in SE^*(3)$, and let α be as in Lemma 4.2. There exists $C>0$ such that for $P \in \mathscr{P}_n$ and n large enough,*

$$|(PW)(x)| \left|1-\left(\frac{x}{a_{4n}}\right)^2\right|^{\alpha-1} \leqslant C_{21} T(a_n)^{1/2} |(PW)(x)| \left|1-\left(\frac{x}{a_{4n}}\right)^2\right|^{\alpha-1/2}, \tag{4.13}$$

uniformly for $|x| \leqslant a_{3n}$.

Proof. This follows from (4.10).

LEMMA 4.4. *Let $W \in SE^*(3)$, $\eta \geqslant a_n$, and define*

$$W_\alpha(x) := W(x)\left(1-\left(\frac{x}{\eta}\right)^2\right)^\alpha, \qquad |x|<\eta, \quad \alpha>0 \text{ fixed.} \tag{4.14}$$

Let $\hat{a}_n$ be the Mhaskar–Rahmanov–Saff number for

$$Q_\alpha(x) = Q(x) - \alpha \log\left(1-\left(\frac{x}{\eta}\right)^2\right).$$

Then

$$a_n \geqslant \hat{a}_n, \tag{4.15}$$

and for any $P \in \mathscr{P}_n$, $n = 1, 2, 3, \ldots$,

$$\max_{[-\eta,\eta]} \left|P(x)\, W(x)\left(1-\left(\frac{x}{\eta}\right)^2\right)^\alpha\right| = \max_{[-a_n,a_n]} \left|P(x)\, W(x)\left(1-\left(\frac{x}{\eta}\right)^2\right)^\alpha\right|. \tag{4.16}$$

Proof. For $0<x<\eta$,

$$Q'_\alpha(x) = Q'(x) + \frac{2(\alpha x/\eta^2)}{1-(x/\eta)^2} \geqslant Q'(x).$$

Then

$$n \geqslant \frac{2}{\pi} \int_0^1 \hat{a}_n t Q'(\hat{a}_n t)(1-t^2)^{-1/2}\, dt,$$

and so,

$$\int_0^1 a_n t Q'(a_n t)(1-t^2)^{-1/2}\, dt \geqslant \int_0^1 \hat{a}_n t Q'(\hat{a}_n t)(1-t^2)^{-1/2}\, dt.$$

Using the inequality above and the fact that $RtQ'(Rt)$ increases as Rt increases, (4.15) follows. By (4.15) and Theorem 2.2(c) and Example 3.3 in [7], (4.16) follows. ∎

LEMMA 4.5. *Let $W \in SE^*(3)$. Fix $\alpha \geq \frac{1}{2}$. Then there exists $C > 0$ such that for $P \in \mathscr{P}_n$ and n large enough,*

$$\max_{[-a_{3n}, a_{3n}]} \left| P'(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha} \right|$$
$$\leq C \frac{n}{a_n} \max_{x \in \mathbb{R}} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right|. \tag{4.17}$$

Proof. Let R_n and α be as in Lemma 4.2. By (4.1), (4.8), and (4.11), and for $|x| \leq a_{3n}$,

$$\left| P'(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha} \right|$$
$$= \left| P'(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right| \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{1/2}$$
$$\sim \left| (P'R_n)(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{1/2} \right|$$
$$\leq \left| (PR_n)'(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{1/2} W(x) \right|$$
$$+ |R_n'(x)| \left| (PW)(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{1/2} \right|$$
$$\leq C_{22} \left\{ \frac{n}{a_n} \max_{x \in \mathbb{R}} |(PR_n)(x)\, W(x)| + \frac{1}{a_{4n}} \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1} |P(x)\, W(x)| \right\}$$
$$\leq C_{23} \left\{ \frac{n}{a_n} \max_{[-a_{4n}, a_{4n}]} |P(x) R_n(x)\, W(x)| \right.$$
$$\left. + \frac{T(a_n)^{1/2}}{a_{4n}} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right| \right\} \qquad \text{(by (4.13) and (2.5))}$$
$$\leq C_{24} \left\{ \frac{n}{a_n} \max_{[-a_n, a_n]} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right| \right\},$$

by (4.16), (4.8), (2.9), and (2.10). Then (4.17) follows from the inequality above.

LEMMA 4.6. *Let $W \in SE^*(3)$. Fix $\alpha \geqslant \frac{1}{2}$. Then there exists $C > 0$ such that for $|x| \geqslant a_{3n}$, $P \in \mathscr{P}_n$, and n large enough,*

$$|P'(x)\, W(x)| \leqslant C \left(\frac{a_{2n}}{|x|}\right)^n \frac{n}{a_n} T(a_n)^{\alpha} \max_{x \in \mathbb{R}} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right|. \tag{4.18}$$

Proof. By (5.1) and (5.2) of Theorem 5.1 in [5], and for $|x| \geqslant a_{2n}$,

$$\begin{aligned} |P'(x)\, x^n W(x)| &\leqslant \max_{[-a_{2n},\, a_{2n}]} |P'(x)\, x^n W(x)| \\ &\leqslant a_{2n}^n \max_{[-a_{2n},\, a_{2n}]} |P'(x)\, W(x)| \\ &\leqslant C_{25} a_{2n}^n \frac{n}{a_n} T(a_n)^{1/2} \|PW\|_{L_\infty(\mathbb{R})} \qquad \text{by (1.18)} \\ &= C_{25} a_{2n}^n \frac{n}{a_n T(a_n})^{1/2} \|PW\|_{[-a_n,\, a_n]}. \end{aligned}$$

Then (4.18) follows since for $|x| \leqslant a_{2n}$,

$$\left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \geqslant \left| 1 - \left(\frac{a_{2n}}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \geqslant C_{26} T(a_n)^{-(\alpha - 1/2)},$$

and so

$$\begin{aligned} T(a_n)^{1/2} \|PW\|_{[-a_n,\, a_n]} &\leqslant C_{26} T(a_n)^{\alpha} \left\| PW \left(1 - \left(\frac{x}{a_{4n}}\right)^2\right)^{\alpha - 1/2} \right\|_{[-a_n,\, a_n]} \\ &\leqslant C_{27} T(a_n)^{\alpha} \left\| PW \left(1 - \left(\frac{x}{a_{4n}}\right)^2\right)^{\alpha - 1/2} \right\|_{L_\infty(\mathbb{R})}, \end{aligned}$$

by (4.16). ∎

LEMMA 4.7. *Let $W \in SE^*(3)$. Fix $\alpha \geqslant \frac{1}{2}$. Then there exists $C > 0$ such that for $|x| \geqslant a_{3n}$, $P \in \mathscr{P}_n$, and n large enough,*

$$\begin{aligned} &\left| P'(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha} \right| \\ &\qquad \leqslant C \frac{n}{a_n} \max_{x \in \mathbb{R}} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right|. \end{aligned} \tag{4.19}$$

Proof. By (4.18) and (2.10), and for $|x| \geqslant a_{3n}$,

$$\begin{aligned}
&\left| P'(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^\alpha \right| \\
&\leqslant C_{28} \left| 1 + \left(\frac{x}{a_{4n}}\right)^2 \right|^\alpha \left(\frac{a_{2n}}{|x|}\right)^n \frac{n}{a_n} T(a_n)^\alpha \\
&\quad \times \max_{x \in \mathbb{R}} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right| \\
&\leqslant C_{29} \left[\left(\frac{a_{2n}}{a_{3n}}\right)^2 + \left(\frac{a_{2n}}{a_{4n}}\right)^2 \right]^\alpha \left(\frac{a_{2n}}{a_{3n}}\right)^{n - 2\alpha} \frac{n}{a_n} T(a_n)^\alpha \\
&\quad \times \max_{x \in \mathbb{R}} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right| \\
&\leqslant C_{30} \frac{n}{a_n} \left(\frac{a_{2n}}{a_{3n}}\right)^{n - 2\alpha} T(a_n)^\alpha \\
&\quad \times \max_{x \in \mathbb{R}} \left| P(x)\, W(x) \left| 1 - \left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha - 1/2} \right|. \qquad (4.20)
\end{aligned}$$

It suffices to show that

$$\left(\frac{a_{2n}}{a_{3n}}\right)^{n - 2\alpha} T(a_n)^\alpha \to 0, \qquad n \to \infty. \qquad (4.21)$$

By (2.9),

$$\begin{aligned}
&\left(\frac{a_{2n}}{a_{3n}}\right)^{n - 2\alpha} T(a_n)^\alpha \\
&\quad = \exp\left[(n - 2\alpha) \log\left(\frac{a_{2n}}{a_{3n}}\right) + \alpha \log T(a_n) \right] \\
&\quad \leqslant \exp\left[(n - 2\alpha)(-C_{31}/T(a_n)) + \alpha \log T(a_n) \right] \\
&\quad = \exp\left\{ \frac{n}{T(a_n)} \left[-C_{31} \left(1 - \frac{2\alpha}{n}\right) + \alpha \left(\frac{\log T(a_n)}{n}\right) T(a_n) \right] \right\}.
\end{aligned}$$

Taking limits as $n \to \infty$ and using (2.11), we get (4.21), and so (4.19) follows from (4.20) and (4.21). ∎

THEOREM 4.8. *Let $W \in SE^*(3)$. Fix $\alpha \geqslant \frac{1}{2}$. Then there exists $C > 0$ such that for $P \in \mathscr{P}_n$ and n large enough,*

$$\max_{x\in\mathbb{R}} \left| P'(x)\, W(x) \left| 1-\left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha} \right|$$
$$\leqslant C\frac{n}{a_n} \max_{x\in\mathbb{R}} \left\| P(x)\, W(x) \left| 1-\left(\frac{x}{a_{4n}}\right)^2 \right|^{\alpha-1/2} \right|. \tag{4.22}$$

Proof. Equation (4.22) follows from (4.17) and (4.19). ∎

LEMMA 4.9. *Fix* $\alpha, \Delta, \eta > 0$. *Then there exists* $C>0$ *such that for all* $x \in \mathbb{R}$,

$$\left|1-\left(\frac{x}{a_{\Delta n}}\right)^2\right|^{\alpha} \leqslant C\left[\left|1-\left(\frac{x}{a_{\eta n}}\right)^2\right|^{\alpha} + T(a_n)^{-\alpha}\right]. \tag{4.23}$$

Proof. By (2.9),

$$\left|1-\left(\frac{x}{a_{\Delta n}}\right)^2\right|^{\alpha} = \left|\left(1-\left(\frac{x}{a_{\eta n}}\right)^2\right) + \left(\left(\frac{x}{a_{\eta n}}\right)^2 - \left(\frac{x}{a_{\Delta n}}\right)^2\right)\right|^{\alpha}$$
$$\leqslant C_{32}\left|1-\left(\frac{x}{a_{\eta n}}\right)^2\right|^{\alpha} + \left|\frac{x}{a_{\eta n}}\right|^{2\alpha} T(a_n)^{-\alpha}.$$

If $|x| \leqslant 2\max\{a_{2\eta n}, a_{2\Delta n}\}$, then

$$\left|\frac{x}{a_{\eta n}}\right| \leqslant 2\max\left\{\frac{a_{2\eta n}}{a_{\eta n}}, \frac{a_{2\Delta n}}{a_{\eta n}}\right\} \leqslant C_{32}.$$

So, (4.23) follows in this case from the two inequalities above. If $|x| \geqslant 2\max\{a_{2\eta n}, a_{2\Delta n}\}$, then $|x| \geqslant a_{\Delta n}$, and so

$$\left|1-\left(\frac{x}{a_{\Delta n}}\right)^2\right|^{\alpha} \leqslant \left|\frac{x}{a_{\Delta n}}\right|^{2\alpha}$$

while

$$\left|1-\left(\frac{x}{a_{\eta n}}\right)^2\right|^{\alpha} = \left|\frac{x}{a_{\eta n}}\right|^{2\alpha} \left|1-\left(\frac{a_{\eta n}}{x}\right)^2\right|^{\alpha}$$
$$\geqslant \left|\frac{x}{a_{\eta n}}\right|^{2\alpha}\left(1-\frac{1}{4}\right)^{\alpha} = C_{33}\left|\frac{x}{a_{\eta n}}\right|^{2\alpha}.$$

These two inequalities together imply

$$\left|1-\left(\frac{x}{a_{\Delta n}}\right)^2\right|^{\alpha} \leqslant C_{34}\left|1-\left(\frac{x}{a_{\eta n}}\right)^2\right|^{\alpha},$$

and so (4.23) holds for such x. ∎

LEMMA 4.10. *Let* $\varphi_n(x) := |1-x^2| + 1/T(a_n)$, $x \in \mathbb{R}$. *Fix* $\beta \geqslant 0$, $\delta > 0$. *Then*

$$\varphi_n\left(\frac{x}{a_{\delta n}}\right)^\beta \sim 1/T(a_n)^\beta + \left|1-\left(\frac{x}{a_{\delta n}}\right)^2\right|^\beta \quad \text{for all} \quad x \in \mathbb{R}. \tag{4.24}$$

Proof. Obvious. ∎

Proof of (1.16) *of Theorem* 1.5. Fix $\beta, \Delta > 0$. By (4.22), (4.23), and (1.18),

$$\begin{aligned}
&\left|P'(x)\, W(x)\, \varphi_n\left(\frac{x}{a_{\beta n}}\right)^\alpha\right| \\
&\quad \leqslant C_{35}\left\{|P'(x)\, W(x)| \left|1-\left(\frac{x}{a_{4n}}\right)^2\right|^\alpha + T(a_n)^{-\alpha}\, |P'(x)\, W(x)|\right\} \\
&\quad \leqslant C_{36}\left\{\frac{n}{a_n} \max_{x\in\mathbb{R}} |P(x)\, W(x)| \left|1-\left(\frac{x}{a_{4n}}\right)^2\right|^{\alpha-1/2}\right. \\
&\qquad \left. + T(a_n)^{-(\alpha-1/2)}\frac{n}{a_n}\, \|PW\|_{L_\infty(\mathbb{R})}\right\} \\
&\quad \leqslant C_{37}\, \frac{n}{a_n} \max_{x\in\mathbb{R}} |P(x)W(x)| \left[\left|1-\left(\frac{x}{a_{\Delta n}}\right)^2\right|^{\alpha-1/2} + 1/T(a_n)^{\alpha-1/2}\right],
\end{aligned}$$

and so (1.16) follows from (4.25) above. ∎

Proof of (1.17) *of Theorem* 1.6. Fix $\beta > 1$. By (4.23), (4.22), and (1.19),

$$\begin{aligned}
&\max_{x\in\mathbb{R}}\left|P'(x)\, W(x)\left|1-\left(\frac{x}{a_{\beta n}}\right)^2\right|^\alpha\right| \\
&\quad \leqslant C_{38}\left\{\max_{x\in\mathbb{R}}\left|P'(x)\, W(x)\left|1-\left(\frac{x}{a_{4n}}\right)^2\right|^\alpha\right| + T(a_n)^{-\alpha} \max_{x\in\mathbb{R}} |P'(x)\, W(x)|\right\} \\
&\quad \leqslant C_{39}\, \frac{n}{a_n}\left\{\max_{x\in\mathbb{R}}\left|P(x)\, W(x)\left|1-\left(\frac{x}{a_{4n}}\right)^2\right|^{\alpha-1/2}\right|\right. \\
&\qquad \left. + T(a_n)^{-(\alpha-1/2)}\, \|PW\|_{L_\infty(\mathbb{R})}\right\} \\
&\quad \leqslant C_{40}\, \frac{n}{a_n}\left\{\max_{x\in\mathbb{R}}\left|P(x)\, W(x)\left|1-\left(\frac{x}{a_{\beta n}}\right)\right|^{\alpha-1/2}\right|\right. \\
&\qquad \left. + T(a_n)^{-(\alpha-1/2)}\, \|PW\|_{[-a_n, a_n]}\right\},
\end{aligned}$$

and so (1.17) follows from (2.9) since $|1-(x/a_{\beta n})^2| \geqslant 1-(a_n/a_{\beta n})^2 \geqslant C/T(a_n)$ for $|x| \leqslant a_n$. ∎

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